

# 4d/5d Correspondence for the Black Hole Potential and its Critical Points

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## Abstract

We express the  $d = 4$ ,  $\mathcal{N} = 2$  black hole effective potential for cubic holomorphic  $F$  functions and generic dyonic charges in terms of  $d = 5$  real special geometry data. The  $4d$  critical points are computed from the  $5d$  ones, and their relation is elucidated. For symmetric spaces, we identify the BPS and non-BPS classes of attractors and the respective entropies. These always derive from simple interpolating formulæ between four and five dimensions, depending on the volume modulus and on the  $4d$  magnetic (or electric) charges.

# 1 Introduction

Recently there has been an increasing amount of work on extremal charged black holes in an environment of scalar background fields, as they naturally arise in modern theories of gravity: superstrings,  $M$ -theory, and their low-energy description through supergravity. In particular, the Attractor Mechanism for extremal black holes [1, 2, 3, 4, 5] in four and five dimensions has been widely investigated for both  $\mathcal{N} = 2$  and extended supergravities [6]–[44] (see also [45], [46] and [47] for recent reviews).

The latest studies on BPS and non-BPS attractor points have developed along two main lines.

In a top down approach [26, 40, 43, 48, 49] one uses some powerful group theoretical techniques, descending from the geometric properties of moduli spaces and  $U$ -duality invariants [50, 51, 52, 53, 54], to classify the solutions to the attractor equations and their properties. For theories with a symmetric scalar manifold, these methods have led to a general classification of BPS and non-BPS attractors, as well as studies of their classical stability and entropy [26, 40, 43, 48, 49]. For  $\mathcal{N} = 2$ ,  $d = 4$  theories, some of these results have also been extended to more general scalar manifolds based on cubic holomorphic prepotentials. These so-called special Kähler “ $d$ -geometries” [55] are particularly relevant, as they naturally arise in the large volume limit of Calabi-Yau compactifications of type IIA superstrings. They include all special Kähler coset manifolds  $G/H$ , that contain symmetric spaces are a further subclass. Moreover, in  $\mathcal{N} = 2$ ,  $d = 4$  supergravity, cubic geometries are precisely those that can be uplifted to five dimensions. Indeed, all (rank-3) symmetric special Kähler spaces fall into this class, and they admit a  $d = 5$  uplift to (rank-2) symmetric real special spaces [55, 56, 57, 58, 59, 60, 61]. We remind that a generic  $d$ -geometry of complex dimension  $n$  is not necessarily a coset space, but nevertheless it has  $n + 1$  isometries, corresponding to the shifts of the  $n$  axions, and to an overall rescaling of the prepotential (see *e.g.* [55] and Refs. therein).

Conversely, in a bottom up approach, one attempts to construct solutions (BH, magnetic strings, black rings, etc.) for a given background spacetime geometry, by solving explicitly the equations of motion [5, 6, 7, 15, 28, 62, 63]. These are originally the second order differential field equations for the scalars and the warp factors, but they have been shown to be equivalent to first order flow equations for both supersymmetric and broad classes of non-supersymmetric, static and rotating BH solutions [36, 42, 44]. In this context, the relation between five and four dimensions is implemented through dimensional reduction and by a Taub-NUT geometry for the black hole (see for instance [8, 9, 11, 44, 64]).

This note brings closer these two lines of analysis and aims at further exploiting, in the top down approach based on the  $4d$  black-hole effective potential, the 5-dimensional origin of  $\mathcal{N} = 2$ , special Kähler  $d$ -geometries. To this end, we first write down the effective black hole potential in terms of the 5d real special geometry data, for generic dyonic charges and scalar field values. Then we proceed to extremisation of this potential with respect to the moduli, and we characterize the attractor points in a 5d language. This will be achieved by connecting the critical points of the BH effective potential through the interpolating formulæ between four and five dimensions. Furthermore, we derive the corresponding entropies which, for symmetric scalar manifolds, are known to be given by the cubic and quartic invariants of the  $U$ -duality groups, built solely in terms of the bare electric and/or magnetic charges of the given BH configuration [50, 51, 52, 53, 54]. Notice that, compared to previous literature such as [11, 44, 64], most of our formulæ hold for generic points in moduli space, and/or that they cover both BPS and non-BPS solutions.

The paper is organized as follows. In Section 2 we recall some formulæ and results holding for special  $d$ -geometries of  $\mathcal{N} = 2$ ,  $d = 4$  and  $d = 5$  supergravity. To this end, we work in the basis of special coordinates, as they are those that naturally provide the link to five dimensions. Thence, we compute the effective potential  $V$  for generic dyonic charges and  $d$ -geometry, and we give its properties for specific BH charge configuration where it undergoes some simplifications. In Section 3 we give  $4d$  attractors in terms of  $5d$  ones, and we compute the corresponding BH entropy. For symmetric spaces, the interpolation yields a clear relation between the  $\mathcal{N} = 2$ ,  $d = 4$  and  $d = 5$   $\frac{1}{2}$ -BPS and non-BPS BH charge orbits studied in the literature [26, 40, 43, 48, 49], which is developed in Section 4. Finally, some further comments and outlooks are given in Section 5.

## 2 Special Geometry for Cubic Holomorphic Prepotentials

We consider  $\mathcal{N} = 2$ ,  $d = 4$  special Kähler geometry based, in special coordinates  $X^\Lambda = (X^0, X^0 z^i)$ , on the holomorphic prepotential

$$F(X) = \frac{1}{3!} d_{ijk} \frac{X^i X^j X^k}{X^0} = (X^0)^2 f(z);$$

$$f(z) \equiv \frac{1}{3!} d_{ijk} z^i z^j z^k.$$
(2.1)

In the Kähler gauge  $X^0 \equiv 1$  and in the special coordinate basis, the Kähler potential reads ( $f_i \equiv \frac{\partial f(z)}{\partial z^i} = \frac{1}{2} d_{ijk} z^j z^k$ )

$$K = -\ln(Y);$$

$$Y \equiv i \left[ 2(f - \bar{f}) + (\bar{z}^i - z^i)(f_i + \bar{f}_{\bar{i}}) \right] = -\frac{i}{3!} d_{ijk} (z^i - \bar{z}^i)(z^j - \bar{z}^j)(z^k - \bar{z}^k). \quad (2.2)$$

By defining the real components of the  $d = 4$  moduli as  $z^i = x^i - i\lambda^i$ , one gets for the Kähler potential

$$K = -\ln(8\mathcal{V}), \quad \mathcal{V} \equiv \frac{1}{3!} d_{ijk} \lambda^i \lambda^j \lambda^k, \quad (2.3)$$

and therefore also the Kähler metric  $g_{i\bar{j}} \equiv \partial_i \partial_{\bar{j}} K$  becomes a real function of the  $\lambda^i$  variables only [66],

$$g_{i\bar{j}} \equiv g_{ij} = -\frac{3}{2} \left( \frac{\kappa_{ij}}{\kappa} - \frac{3}{2} \frac{\kappa_i \kappa_j}{\kappa^2} \right) = -\frac{1}{4} \frac{\partial^2 \ln(\mathcal{V})}{\partial \lambda^i \partial \lambda^j} \Leftrightarrow g^{i\bar{j}} = 2 \left( \lambda^i \lambda^j - \frac{\kappa}{3} \kappa^{ij} \right) \equiv g^{ij};$$

$$\kappa_{ij} \equiv d_{ijk} \lambda^k, \quad \kappa_i \equiv d_{ijk} \lambda^j \lambda^k, \quad \kappa \equiv d_{ijk} \lambda^i \lambda^j \lambda^k = 6\mathcal{V}, \quad \kappa^{ij} \kappa_{jl} \equiv \delta_l^i; \quad (2.4)$$

We introduce the  $d = 5$  real moduli as  $\lambda^i \equiv \mathcal{V}^{1/3} \hat{\lambda}^i$ . They satisfy

$$\frac{1}{3!} d_{ijk} \hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k = 1, \quad (2.5)$$

which is the defining equation of the  $d = 5$  real special manifold. In these variables one gets

$$g_{ij} = \frac{1}{4} \left( \frac{1}{4} \hat{\kappa}_i \hat{\kappa}_j - \hat{\kappa}_{ij} \right) \mathcal{V}^{-2/3} = \frac{1}{4} \mathcal{V}^{-2/3} a_{ij} \Leftrightarrow g^{ij} = 2 \left( \hat{\lambda}^i \hat{\lambda}^j - 2 \hat{\kappa}^{ij} \right) \mathcal{V}^{2/3} = 4 \mathcal{V}^{2/3} a^{ij};$$

$$\hat{\kappa}_{ij} \equiv d_{ijk} \hat{\lambda}^k, \quad \hat{\kappa}_i \equiv d_{ijk} \hat{\lambda}^j \hat{\lambda}^k, \quad \hat{\kappa} \equiv d_{ijk} \hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k = 6, \quad \hat{\kappa}^{ij} \hat{\kappa}_{jl} \equiv \delta_l^i, \quad (2.6)$$

where

$$a_{ij} \equiv 4g_{ij}|_{\mathcal{V}=1}, \quad a^{ij} a_{jk} = \delta_k^i \Rightarrow a_{ij} \hat{\lambda}^j = \frac{1}{2} \hat{\kappa}_i, \quad a_{ij} \hat{\lambda}^i \hat{\lambda}^j = 3. \quad (2.7)$$

One can then proceed to computing also the vector kinetic matrix  $\mathcal{N}_{\Lambda\Sigma}$  in terms of these quantities, obtaining

$$Im\mathcal{N}_{\Lambda\Sigma} = -\mathcal{V} \begin{pmatrix} 1+4g & g_j \\ g_i & 4g_{ij} \end{pmatrix} \Leftrightarrow (Im\mathcal{N}_{\Lambda\Sigma})^{-1} = -\frac{1}{\mathcal{V}} \begin{pmatrix} 1 & x^j \\ x^i & x^i x^j + \frac{1}{4} g^{ij} \end{pmatrix};$$

$$Re\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \frac{1}{3} h & -\frac{1}{2} h_j \\ -\frac{1}{2} h_i & h_{ij} \end{pmatrix}, \quad (2.8)$$

where

$$\begin{aligned} g &\equiv g_{ij}x^ix^j, \quad g_i \equiv -4g_{ij}x^j; \\ h_{ij} &\equiv d_{ijk}x^k, \quad h_i \equiv d_{ijk}x^jx^k, \quad h \equiv d_{ijk}x^ix^jx^k. \end{aligned} \quad (2.9)$$

Note that in the above special coordinate basis, the index 0, associated to the graviphoton vector, is naturally split from the  $d = 5$  index  $i = 1, \dots, n_V$ . In this language, the  $d = 5$  real special manifold is simply the  $(n_V - 1)$  real hypersurface with unit volume. Accordingly, the  $n_V$   $d = 4$  complex moduli  $z^i$  separate into  $(x^i, \mathcal{V}, \hat{\lambda}^i)$ , where  $\hat{\lambda}^i$  are the  $n_V$  real positive  $d = 5$  moduli, parameterizing the hypersurface  $\frac{1}{3!}d_{ijk}\hat{\lambda}^i\hat{\lambda}^j\hat{\lambda}^k = 1$ .

Let us now consider the  $d = 4$  BH effective potential,

$$V = -\frac{1}{2}Q^T\mathcal{M}Q, \quad (2.10)$$

where  $Q$  denotes the  $(2n_V + 2)$  charge vector

$$Q = (p^0, p^i, q_0, q_i), \quad (2.11)$$

and  $\mathcal{M}$  is the real symplectic  $(2n_V + 2) \times (2n_V + 2)$  matrix

$$\mathcal{M} \equiv \begin{pmatrix} \text{Im}\mathcal{N} + \text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1}\text{Re}\mathcal{N} & -\text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1} \\ -(\text{Im}\mathcal{N})^{-1}\text{Re}\mathcal{N} & (\text{Im}\mathcal{N})^{-1} \end{pmatrix}. \quad (2.12)$$

By using Eqs. (2.10)-(2.12) and the expressions computed in Eq. (2.8), one obtains the following formula of the  $d = 4$  effective BH potential for a generic special Kähler  $d$ -geometry and dyonic charges:

$$\begin{aligned} 2V = & \left[ \frac{\kappa}{6}(1 + 4g) + \frac{h^2}{6\kappa} + \frac{3}{8\kappa}g^{ij}h_ih_j \right] (p^0)^2 + \\ & + \left[ \frac{2}{3}\kappa g_{ij} + \frac{3}{2\kappa}(h_ih_j + h_{im}g^{mn}h_{nj}) \right] p^ip^j + \\ & + \frac{6}{\kappa} \left[ (q_0)^2 + 2x^iq_0q_i + \left( x^ix^j + \frac{1}{4}g^{ij} \right) q_iq_j \right] + \\ & + 2 \left[ \frac{\kappa}{6}g_i - \frac{h}{2\kappa}h_i - \frac{3}{4\kappa}g^{jm}h_mh_{ij} \right] p^0p^i + \\ & - \frac{2}{\kappa} \left[ -hp^0q_0 + 3q_0p^ih_i - \left( hx^i + \frac{3}{4}g^{ij}h_j \right) p^0q_i + 3 \left( h_jx^i + \frac{1}{2}g^{im}h_{mj} \right) q_ip^j \right]. \end{aligned} \quad (2.13)$$

A quick look reveals that the axions  $x^i$  appear in  $V$  through a polynomial of degree 6, whose coefficients depend on  $\lambda^i$  and on  $d_{ijk}$ . Moreover, terms linear

in  $x^i$  vanish if  $q_0 q_i$ ,  $p^0 p^i$  and  $p^i q_j$  separately vanish. This means that for BH charge configurations of the type

$$\begin{aligned} a) \quad Q_0 &= (p^0, 0, q_0, 0); \\ b) \quad Q_e &= (p^0, 0, 0, q_i); \\ c) \quad Q_m &= (0, p^i, q_0, 0). \end{aligned} \tag{2.14}$$

the Attractor Equations  $\frac{\partial V}{\partial x^i} = 0$  admit the  $n$  solutions given by  $x^i = 0$ , *i.e.* by purely imaginary critical moduli  $z^i = -i\lambda^i$ .

## 2.1 Symmetric $d$ -geometries

Something more can be said if one considers symmetric space theories associated to scalar manifolds  $G/H$ , such that  $G$  is a symmetry of the action. The discussion below will rely on the results of [49, 48, 26]. All symmetric special Kähler  $d$ -geometries are known to originate from five dimensional theories through dimensional reduction. In this case, the tensor  $d_{ijk}$  is an invariant tensor of the  $U$ -duality  $d = 5$  group  $G_5$ , and  $\lambda^i$  (and  $x^i$ ) transform linearly under  $G_5$  (we recall that  $G_4$  decomposes into  $G_5 \otimes SO(1, 1)$ , where  $SO(1, 1)$  corresponds to the radius of compactifications along  $S^1$  [57]). Furthermore, the symmetric tensor  $d_{ijk}$  satisfies the non-linear relation [57, 60]

$$d_{r(pq} d_{ij)k} d^{rkl} = \frac{4}{3} \delta_{(p}^l d_{qij)}, \tag{2.15}$$

which is equivalent to the condition for the corresponding manifold (in  $d = 4$  and  $d = 5$ ) to be symmetric.

For a symmetric real special manifold  $\frac{G_5}{H_5}$ , one can always perform a suitable  $H_5$ -transformation that brings the cubic polynomial to *normal form*,

$$I_3(q) = \frac{1}{3!} d^{ijk} q_i q_j q_k = q_1 q_2 q_3, \tag{2.16}$$

where  $q_1$ ,  $q_2$  and  $q_3$  are the three eigenvalues of the corresponding  $3 \times 3$  Jordan system. For non-symmetric  $d$ -geometries Eq. (2.16) does not hold any more, but nevertheless we will confine our discussion of  $d = 5$  to the 3-charge case. The simplest example of this kind is provided by the 5-dimensional uplift of the *stu* model [69, 62, 28], consisting in a 2-dimensional free  $\sigma$ -model  $SO(1, 1) \otimes SO(1, 1)$ , whose real special geometry is determined by the constraint

$$\frac{1}{3!} d_{ijk} \hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k = \hat{\lambda}^1 \hat{\lambda}^2 \hat{\lambda}^3 = 1. \tag{2.17}$$

This is the model we will use to perform most of the computations, even though our results will hold in general for rank-2 symmetric real special manifolds.

We now consider more in depth the BH charge configurations in (2.14) for symmetric  $d$ -geometries. In this case, the quartic invariant is given by (see [54] for notation and further elucidation)

$$I_4(p^0, p^i, q_0, q_i) = - (p^0 q_0 + p^i q_i)^2 + 4 \left[ q_0 I_3(p) - p^0 I_3(q) + \left\{ \frac{\partial I_3(p)}{\partial p}, \frac{\partial I_3(q)}{\partial q} \right\} \right] \quad (2.18)$$

in terms of the cubic invariants of the five-dimensional U-duality group  $G_5$ , with

$$I_3(p) = \frac{1}{3!} d_{ijk} p^i p^j p^k, \quad \{I_3(q), I_3(p)\} \equiv \frac{\partial I_3(q)}{\partial q_i} \frac{\partial I_3(p)}{\partial p^i}. \quad (2.19)$$

Note that all terms of Eq. (2.18) are invariant under  $G_5$ , because  $p^i$ 's and  $q_i$ 's transform as the linear gradient and contragradient representation of  $G_5$ , respectively.

According to the classification of charge orbits for symmetric  $d$ -geometries, it is known that for  $d = 5$  there are two distinct orbits (one BPS and the other non-BPS) [49, 48], whereas for  $d = 4$  there exist three orbits, one BPS and two non-BPS [26]. For a given BH charge configuration, the  $d = 5$  and  $d = 4$  charge orbits, respectively with 3 and 4 distinct eigenvalues, can actually cover all cases, as follows [49, 48, 26]:

$$d = 5 : \begin{cases} \text{BPS} : (+ + +) \text{ or } (- - -); \\ \text{non-BPS} : (+ + -) \text{ or } (- - +); \end{cases} \quad (2.20)$$

$$d = 4 : \begin{cases} \text{BPS} : (+ + + +) \text{ or } (- - - -); \\ \text{non-BPS, } Z \neq 0 : (+ + + -) \text{ or } (- - - +); \\ \text{non-BPS, } Z = 0 : (+ + - -). \end{cases}$$

For the BH charge configurations in (2.14), one gets

$$\begin{aligned} a) \quad & I_4(p^0, q_0) = - (p^0 q_0)^2; \\ b) \quad & I_4(p^0, q_i) = -4p^0 I_3(q); \\ c) \quad & I_4(p^i, q_0) = 4q_0 I_3(p). \end{aligned} \quad (2.21)$$

While the configuration a),  $(p^0, q_0)$ , clearly gives only  $I_4 < 0$ , the other two configurations b) and c), which are reciprocally dual, yield  $I_4 \geq 0$ , depending on whether the cubic invariant  $I_3(q)$  and  $I_3(p)$  have the same or opposite sign of  $p^0$  or  $q_0$ . In the subsequent treatment of symmetric spaces we will consider only one of these charge configurations, namely the *electric* one associated to  $Q_e$ .

Let us take the  $d = 5$  ( $\frac{1}{2}$ -)BPS configuration where  $q_1, q_2, q_3$  all have positive sign, so that  $I_3(q) > 0$ . It is obvious that, depending on the sign of  $p^0$  one will get a  $d = 4$  ( $\frac{1}{2}$ -)BPS or non-BPS  $Z \neq 0$  configuration. Indeed, since  $I_3(q) > 0$  implies that  $I_4(p^0, q_i) \geq 0$  according to  $p^0 \leq 0$ , one gets that  $q_1, q_2, q_3 > 0$  is the  $d = 4$  BPS configuration, where we identify  $(-p^0, q_1, q_2, q_3)$  with the 4 positive eigenvalues of the corresponding Freudenthal triple system.

On the other hand, if one starts with a  $d = 5$  non-BPS configuration, say with  $q_1, q_2 < 0$  and  $q_3 > 0$ , then, for either sign of  $p^0$ , one always obtains a  $d = 4$  non-BPS configuration. The sign of  $p^0$  is however crucial, because it gives rise to two inequivalent non-BPS charge configurations, distinguished by the vanishing of the central charge:

$$\begin{aligned} p^0 < 0 &\Rightarrow \text{non-BPS, } Z = 0, \quad I_4 > 0; \\ p^0 > 0 &\Rightarrow \text{non-BPS, } Z \neq 0, \quad I_4 < 0. \end{aligned} \tag{2.22}$$

In the next Section we will study this phenomenon by directly solving the Attractor Equations.

### 3 $5d$ and $4d$ Black Hole Potentials, Entropies and Their Interpolation

In this Section we study the relation between the critical points of  $V_5^q$  and their  $d = 4$  counterparts, for the BH charge configurations in (2.14). More specifically, we will discuss the interpolation between  $d = 5$  and  $d = 4$  extremal BHs in  $\mathcal{N} = 2$  supergravity for some particular cases, namely for the aforementioned BH charge configurations in (2.14). For  $CY_3$ -compactifications these charge configurations respectively correspond to switching on the charges of a)  $D0 - D6$  branes; b)  $D0 - D4$  branes; c)  $D2 - D6$  branes. As previously mentioned, configuration b) is called *electric*, and its uplift to  $d = 5$  describes the so-called extremal Tangherlini BH [67, 68] with horizon geometry  $AdS_2 \times S^3$ . On the other hand, configuration c) is called *magnetic*, and its uplift to  $d = 5$  describes a black string with horizon geometry  $AdS_3 \times S^2$ .



For symmetric  $d$ -geometries configurations  $b)$  and  $c)$  are reciprocally dual, but this does not hold any more for a generic  $d$ -geometry. We will consider only configuration  $b)$  in our treatment, briefly commenting at the end of the Section on the extension of our results to non-symmetric  $d$ -geometries.

### 3.1 $d = 4$ Black Hole Potential for Vanishing Axions

As previously noticed, the  $\mathcal{N} = 2$ ,  $d = 4$  extremal BH potential  $V$ , given for a generic special Kähler  $d$ -geometry by Eq. (2.13), undergoes a dramatic simplification when no linear terms in the axions  $x^i$  appear, such that the criticality conditions  $\frac{\partial V}{\partial x^i} = 0$  can be solved by putting  $x^i = 0 \ \forall i$ . For this situation, we will introduce the notation  $V^* = V|_{x^i=0 \ \forall i}$ .

By further considering  $Q_0 = (p^0, 0, q_0, 0)$  (BH charge configuration  $a$ )), Eq. (2.13) yields

$$V^* = \frac{1}{2} \left[ (p^0)^2 \mathcal{V} + (q_0)^2 \mathcal{V}^{-1} \right] = V^*(\mathcal{V}, p^0, q_0), \quad (3.1)$$

where we have used the relation  $\kappa = 6\mathcal{V}$ . By recalling the redefinition  $\lambda^i \equiv \mathcal{V}^{1/3} \hat{\lambda}^i$ , it is immediate to realize that for the BH charge configuration  $a$ ) the effective BH potential  $V^*$  at vanishing axions *does not depend on any of the*  $\hat{\lambda}^i$ , and thus it has  $n_V - 1$  “flat” directions at all orders.

This result agrees with the findings of [41], and also with the analysis performed in [40] and [43]. Indeed, for symmetric spaces, the moduli space of the  $d = 4$  non-BPS  $Z \neq 0$  orbit ( $I_4 < 0$ ) given in 2.21 coincides with the special real scalar manifold of the  $d = 5$  parent theory [43].

It is easy to realize that this result actually holds for a generic  $d$ -geometry rather than for only symmetric ones. From a  $d = 5$  perspective, no  $5d$  charges are turned on, because  $p^0$  and  $q_0$  are the charges of the Kaluza-Klein vector, and thus one would not expect that the  $5d$  moduli  $\hat{\lambda}^i$  are stabilized, as it actually happens.

Since

$$\frac{\partial V^*}{\partial \mathcal{V}} = 0 \Leftrightarrow \mathcal{V} = \left| \frac{q_0}{p^0} \right|, \quad (3.2)$$

by defining

$$V_{(p^0, q_0), crit.}^* \equiv V^*|_{\frac{\partial V}{\partial \mathcal{V}}=0}, \quad (3.3)$$

one gets

$$\frac{S_{BH}}{\pi} = V_{(p^0, q_0), crit.}^* = |p^0 q_0|. \quad (3.4)$$

It is interesting to observe that Eq. (3.2) and (3.4) are the same ones of the so-called *dilatonic BH* (see [3] and Refs. therein). However, whereas the

dilatonic BH is BPS, the present case has  $I_4 < 0$ , and thus it corresponds to a non-BPS  $Z \neq 0$  attractor.

Let us now move to consider  $Q_e = (p^0, 0, 0, q_i)$  (BH charge configuration  $b$ ), that is in  $d = 5$  a generic charge configuration for the extremal Tangherlini BH. Then Eq. (2.13) yields

$$V^* = \frac{1}{2} \left[ (p^0)^2 \mathcal{V} + \mathcal{V}^{-1/3} a^{ij} q_i q_j \right] = V^* \left( \mathcal{V}, \hat{\lambda}^i, p^0, q_j \right). \quad (3.5)$$

By defining the 5d black hole potential

$$V_5^q \equiv a^{ij} q_i q_j = V_5^q \left( \hat{\lambda}^i, q_j \right), \quad (3.6)$$

one obtains that

$$\frac{\partial V^*}{\partial \hat{\lambda}^i} = 0 \implies \frac{\partial V_5^q}{\partial \hat{\lambda}^i} = 0, \quad \forall i. \quad (3.7)$$

Since

$$\frac{\partial V^*}{\partial \mathcal{V}} = 0 \Leftrightarrow \mathcal{V}^{4/3} = \frac{1}{3(p^0)^2} V_5^q, \quad (3.8)$$

by defining

$$V_{(p^0, q_i), crit.}^* \equiv V^* \quad \text{at} \quad \begin{cases} \frac{\partial V^*}{\partial \mathcal{V}} = 0; \\ \frac{\partial V^*}{\partial \hat{\lambda}^i} = 0 \quad \forall i, \end{cases} \quad (3.9)$$

one gets

$$\frac{S_{BH}}{\pi} = V_{(p^0, q_i), crit.}^* = 2 |p^0|^{1/2} \left( \frac{V_5^q}{3} \right)^{3/4} \bigg|_{\frac{\partial V_5^q}{\partial \hat{\lambda}^i} = 0 \quad \forall i}, \quad (3.10)$$

a formula valid for any  $d$ -geometry (for vanishing axions, and in the BH charge configuration  $Q = (p^0, 0, 0, q_i)$ ).

For a symmetric  $d$ -geometry, it holds that

$$\frac{V_5^q}{3} \bigg|_{\frac{\partial V_5^q}{\partial \hat{\lambda}^i} = 0 \quad \forall i} = \left( \frac{1}{3!} |d^{ijk} q_i q_j q_k| \right)^{2/3} = |q_1 q_2 q_3|^{2/3}, \quad (3.11)$$

so one finally gets

$$\mathcal{V}_{crit.}^{4/3} = \frac{1}{(p^0)^2} \left( \frac{1}{3!} |d^{ijk} q_i q_j q_k| \right)^{2/3} = \frac{|q_1 q_2 q_3|^{2/3}}{(p^0)^2}, \quad (3.12)$$

and

$$\frac{S_{BH}}{\pi} = V_{(p^0, q_i), crit.}^* = 2 |p^0 q_1 q_2 q_3|^{1/2}, \quad (3.13)$$

which is the known  $d = 4$  result [62, 28, 41]. Since the  $d = 5$  BH entropy has the general form (for the electric charge configuration) ([3, 48]; see also Eq. (4.22))

$$\frac{S_{BH,d=5}}{\pi} = \left( 3 V_5^q \Big|_{\frac{\partial V_5^q}{\partial \lambda^i} = 0 \ \forall i} \right)^{3/4}, \quad (3.14)$$

one also obtains that (see also Eq. (4.23))

$$\frac{S_{BH,d=5}}{\pi} = 3^{\frac{3}{2}} |q_1 q_2 q_3|^{1/2}. \quad (3.15)$$

The above formulae may be compared to those obtained in [64] with a different approach, that is in the context of the entropy function formalism (which is known to hold only on shell for the scalar fields), and where vanishing axions are those associated to non-rotating black holes (see also [44]).

We also remark that the use of a five-dimensional BH metric Ansatz implies that, in order for the  $4d$  BH entropy to have the correct form, the  $5d$  charges must be redefined quadratically in terms of the  $4d$  ones (for our case  $Q_i = -p^0 q_i$ , where  $Q_i$  denotes the 5-dimensional charges; see *e.g.* [65]). In our derivation such redefinition does not occur, because our approach directly takes into account the symmetries of the problem<sup>1</sup>.

Finally, the above computations and results are insensitive to the sign of the BH charges. This fact can be understood by noticing that the charges appear quadratically in  $V^*$  specified for the background charge vector  $Q_e = (p^0, 0, 0, q_i)$ . On the other hand, the supersymmetric nature of the solutions (3.10) and (3.13) crucially depend on the sign of the four eigenvalues  $(-p^0, q_1, q_2, q_3)$  of the corresponding Freudenthal triple system (see *e.g.* [26] and Refs. therein). In the next Section, in the framework of the ( $d = 5$  uplift of the) *stu* model, we will see that, depending on the signs of the charges, the attractor configuration determining the entropy (3.13) has the following supersymmetry-preserving features:

$$\begin{aligned} I_4 < 0 : & \text{ non-BPS, } Z \neq 0; \\ I_4 > 0 : & \text{ either } \tfrac{1}{2}\text{-BPS or non-BPS, } Z = 0. \end{aligned} \quad (3.16)$$

## 4 BPS and non-BPS $d = 5$ and $d = 4$ Relations

We now turn to exploring the BPS/non-BPS nature of the 5d/4d attractors by considering the simplest rank-3 special Kähler geometry, that is the

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<sup>1</sup>Let us also notice that in our notation  $p^0$  and  $q_0$  are opposite to those used in [65].

$stu$  model. However, our results clearly hold for all rank-3 symmetric  $d$ -geometries, indeed all containing the  $stu$  model. Thus, we need to consider the homogeneous symmetric manifold  $\left(\frac{SU(1,1)}{U(1)}\right)^3$ , product of three rank-1 cosets. The  $d = 5$  corresponding real special homogeneous symmetric manifold is  $SO(1,1) \otimes SO(1,1) = \mathbb{R}_0^+ \times \mathbb{R}_0^+$ , *i.e.* the rank-2 product of two rank-1 free trivial spaces. Such manifold can be also characterized as the geometrical *locus* ( $i = 1, 2, 3$ )

$$\frac{1}{3!} d_{ijk} \hat{\lambda}^i \hat{\lambda}^j \hat{\lambda}^k = \hat{\lambda}^1 \hat{\lambda}^2 \hat{\lambda}^3 = 1. \quad (4.1)$$

The  $d = 4$  theory has no non-BPS  $Z = 0$  “flat” directions, and 2 non-BPS  $Z \neq 0$  “flat” directions [15, 26, 41, 40, 43] (in the BH charge configuration  $Q_0 = (p^0, 0, q_0, 0)$  they are precisely given by Eq. (4.1)), whereas the  $d = 5$  theory has no non-BPS “flat” directions at all.

Let us start by analyzing the  $d = 5$  BPS attractors for the BH charge configuration  $Q_e = (p^0, 0, 0, q_i)$ . For the  $d = 5$  uplift of the  $stu$  model the matrix  $a^{ij}$  is diagonal:

$$a_{stu}^{ij} = \text{diag} \left( \left( \hat{\lambda}^1 \right)^2, \left( \hat{\lambda}^2 \right)^2, \left( \hat{\lambda}^3 \right)^2 = \frac{1}{\left( \hat{\lambda}^1 \right)^2 \left( \hat{\lambda}^2 \right)^2} \right), \quad (4.2)$$

and thus the  $stu$  electric  $d = 5$  effective BH potential reads

$$V_{5,stu}^q = \left( \hat{\lambda}^1 \right)^2 q_1 + \left( \hat{\lambda}^2 \right)^2 q_2 + \frac{q_3}{\left( \hat{\lambda}^1 \right)^2 \left( \hat{\lambda}^2 \right)^2}. \quad (4.3)$$

Since

$$\frac{\partial V_{5,stu}^q}{\partial \hat{\lambda}^i} = 0, \quad i = 1, 2 \Leftrightarrow \hat{\lambda}^1 \hat{\lambda}^2 = \left( \frac{(q_3)^2}{|q_1 q_2|} \right)^{1/3}, \quad \frac{\hat{\lambda}^1}{\hat{\lambda}^2} = \left| \frac{q_2}{q_1} \right|, \quad (4.4)$$

one obtains

$$\left. \frac{V_{5,stu}^q}{3} \right|_{\frac{\partial V_{5,stu}^q}{\partial \hat{\lambda}^i} = 0, \quad i=1,2} = |q_1 q_2 q_3|^{2/3}, \quad (4.5)$$

as anticipated in Eq. (3.11).

The (real)  $d = 5$ ,  $\mathcal{N} = 2$  central charge (in the *electric* BH charge configuration  $Q_e$ ), is defined as

$$Z_5^q \equiv \hat{\lambda}^i q_i. \quad (4.6)$$

Thus, one obtains that the  $d = 5$  BPS conditions

$$\frac{\partial Z_5^q}{\partial \hat{\lambda}^i} = 0, \quad i = 1, 2 \Leftrightarrow \left( \hat{\lambda}^1 \right)^2 \hat{\lambda}^2 = \frac{q_3}{q_1}, \quad \hat{\lambda}^1 \left( \hat{\lambda}^2 \right)^2 = \frac{q_3}{q_2}, \quad (4.7)$$

are solved only by requiring  $\frac{q_3}{q_1} > 0$  and  $\frac{q_3}{q_2} > 0$ , *i.e.* for  $q_1, q_2$  and  $q_3$  *all* with the same sign. Consequently, with no loss of generality we can conclude that the  $d = 5$  critical point determining the result (4.5) is BPS if  $q_1, q_2, q_3 > 0$ , and it is non-BPS if  $q_1, q_2 < 0$  and  $q_3 > 0$ .

Now, in order to find the relation with the  $d = 4$  attractors, one must compute the (complex)  $d = 4, \mathcal{N} = 2$  central charge  $Z(z, \bar{z}, Q)$  and its co-variant derivatives. By recalling the standard definition in obvious notation,  $Z = \exp(K/2)(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda)$ , going to special coordinates, in the Kähler gauge ( $X^0 \equiv 1$ ), and exploiting the cubic nature of the holomorphic prepotential  $f(z)$  given in (2.1), one gets

$$Z(z, \bar{z}, Q) = e^{K(z, \bar{z})/2} [q_0 + q_i z^i + p^0 f(z) - p^i f_i(z)] \equiv e^{K(z, \bar{z})/2} W(z, Q), \quad (4.8)$$

where  $W$  is the  $d = 4, \mathcal{N} = 2$  holomorphic superpotential. In terms of the real components of the  $d = 4$  moduli as  $z^i = x^i - i\mathcal{V}^{1/3}\hat{\lambda}^i$  one gets the following expressions ( $e^{K(z, \bar{z})/2} = \frac{1}{2\sqrt{2}}\mathcal{V}^{-1/2}$ ):

$$\begin{aligned} Z(x^j, \hat{\lambda}^j, \mathcal{V}, Q) &= \frac{1}{2\sqrt{2}}\mathcal{V}^{-1/2}W(x^j, \hat{\lambda}^j, \mathcal{V}, Q) = \\ &= \frac{1}{2\sqrt{2}}\mathcal{V}^{-1/2} \left[ \begin{aligned} &q_0 + q_i x^i - i\mathcal{V}^{1/3}q_i \hat{\lambda}^i + \\ &+ \frac{p^0}{6}h - i\frac{p^0}{2}\mathcal{V}^{1/3}\hat{\kappa}_{ij}x^i x^j + \\ &- \frac{p^0}{2}\mathcal{V}^{2/3}\hat{\kappa}_i x^i + ip^0\mathcal{V} + \\ &- \frac{p^i}{2}h_i + ip^i\mathcal{V}^{1/3}\hat{\kappa}_{ij}x^j + \frac{p^i}{2}\mathcal{V}^{2/3}\hat{\kappa}_i \end{aligned} \right]; \end{aligned} \quad (4.9)$$

$$\begin{aligned}
D_i Z \left( x^j, \widehat{\lambda}^j, \mathcal{V}, Q \right) &= \frac{1}{2\sqrt{2}} \mathcal{V}^{-1/2} D_i W \left( x^j, \widehat{\lambda}^j, \mathcal{V}, Q \right) = \\
&= \frac{1}{2\sqrt{2}} \mathcal{V}^{-1/2} \left\{ \begin{aligned} &q_i + \frac{p^0}{2} h_i - i p^0 \mathcal{V}^{1/3} \widehat{\kappa}_{ij} x^j - \frac{p^0}{4} \mathcal{V}^{2/3} \widehat{\kappa}_i + \\ &- p^j h_{ij} + i p^j \mathcal{V}^{1/3} \widehat{\kappa}_{ij} + \\ &- \frac{i}{4} \mathcal{V}^{-1/3} \widehat{\kappa}_i \left[ \begin{aligned} &q_0 + q_j x^j - i \mathcal{V}^{1/3} q_j \widehat{\lambda}^j + \\ &+ \frac{p^0}{6} h - i \frac{p^0}{2} \mathcal{V}^{1/3} \widehat{\kappa}_{jk} x^j x^k - \frac{p^0}{2} \mathcal{V}^{2/3} \widehat{\kappa}_j x^j + \\ &- \frac{p^j}{2} h_j + i p^j \mathcal{V}^{1/3} \widehat{\kappa}_{jk} x^k + \frac{p^j}{2} \mathcal{V}^{2/3} \widehat{\kappa}_j \end{aligned} \right] \end{aligned} \right\}. \tag{4.10}
\end{aligned}$$

Eqs. (4.9) and (4.10) are general, but for our purposes we actually need them just for the particular BH charge configuration  $Q_0 = (p^0, 0, 0, q_i)$ . For this charge vector, Eq. (4.10) yields that

$$x^i = 0, \forall i \implies \text{Im}(D_i Z) = 0, \tag{4.11}$$

and moreover, considering the *stu* model,

$$\frac{q_3}{q_1}, \frac{q_3}{q_2} > 0, p^0 < 0 \implies \text{Re}(D_i Z) = 0, \tag{4.12}$$

with  $\widehat{\lambda}^1$  and  $\widehat{\lambda}^2$  given by Eq. (4.7). Thus, since the  $d = 5$  BPS configuration has  $q_1, q_2$  and  $q_3 > 0$ , it is clear that it will determine a  $d = 4$  BPS configuration if the magnetic charge of the Kaluza-Klein vector is negative (*i.e.*  $p^0 < 0$ ), and a  $d = 4$  non-BPS configuration if  $p^0 > 0$ . In order to determine whether this latter  $d = 4$  configuration is non-BPS  $Z \neq 0$  or non-BPS  $Z = 0$ , one has to check the  $d = 4$  central charge. For vanishing axions, it turns out to be purely imaginary (the asterisk denotes the evaluation at vanishing axions treatment):

$$Z^* = \frac{i}{2\sqrt{2}} \left[ p^0 \mathcal{V}^{1/2} - \mathcal{V}^{-1/6} q_i \widehat{\lambda}^i \right]. \tag{4.13}$$

Since at the  $d = 5$  BPS attractors it holds that  $\left( q_i \widehat{\lambda}^i \right)_{d=5, \text{BPS}} = 3 (q_1 q_2 q_3)^{1/3}$ , by using the critical value of  $\mathcal{V}$  given by Eq. (3.12) and considering  $p^0 < 0$ , one gets that ( $p^0 < 0, q_1 > 0, q_2 > 0, q_3 > 0$ )

$$Z_{BPS}^* = -i\sqrt{2} \left( -p^0 q_1 q_2 q_3 \right)^{1/4} \neq 0. \tag{4.14}$$

Thus, the  $d = 4$   $\frac{1}{2}$ -BPS attractor determined by the  $d = 5$   $\frac{1}{2}$ -BPS attractor configuration  $q_1, q_2, q_3 > 0$  by considering  $p^0 < 0$  has the following non-vanishing BH entropy:

$$\frac{S_{BH}}{\pi} = V_{(p^0, q_i), BPS}^* = |Z^*|_{BPS}^2 = 2 (-p^0 q_1 q_2 q_3)^{1/2}. \quad (4.15)$$

On the other hand, by using the critical value of  $\mathcal{V}$  given by Eq. (3.8) and considering  $p^0 > 0$ , one gets that the  $d = 5$   $\frac{1}{2}$ -BPS attractor configuration  $q_1, q_2, q_3 > 0$  also determines a  $d = 4$  non-BPS attractor with  $(p^0 > 0, q_1 > 0, q_2 > 0, q_3 > 0)$

$$Z_{non-BPS}^* = -\frac{i}{\sqrt{2}} (p^0 q_1 q_2 q_3)^{1/4} \neq 0, \quad (4.16)$$

yielding a non-vanishing BH entropy

$$\frac{S_{BH}}{\pi} = V_{(p^0, q_i), non-BPS, Z^* \neq 0}^* = 4 |Z|_{non-BPS, Z^* \neq 0}^2 = 2 (p^0 q_1 q_2 q_3)^{1/2}, \quad (4.17)$$

as found in [26]. In order to derive Eq. (4.17), we have used the standard definition of  $V$  in terms of central charge and matter charges,  $V = |Z|^2 + |D_i Z|^2$  and we have computed the purely real value  $D_i Z$ , also by using the critical value of  $\mathcal{V}$  given by Eq. (3.8).

Let us now move to considering the  $d = 5$  non-BPS attractors for the BH charge configuration  $Q_e = (p^0, 0, 0, q_i)$ . As pointed out above, in this case one can assume without loss of generality that  $q_1, q_2 < 0$ , and  $q_3 > 0$  (indeed violating the  $d = 5$  BPS conditions). This yields  $\left(q_i \hat{\lambda}^i\right)_{d=5, non-BPS} = -|q_1 q_2 q_3|^{1/3}$ .

If  $p^0 < 0$ , one finds that, using the critical value of  $\mathcal{V}$  given by Eq. (3.8), the two terms in Eq. (4.13) reciprocally cancel ( $p^0 < 0, q_1 < 0, q_2 < 0, q_3 > 0$ ):

$$Z^* = 0. \quad (4.18)$$

Thus, the BH charge configuration  $p^0 < 0, q_1, q_2 < 0$ , and  $q_3 > 0$  supports a  $d = 4$  non-BPS  $Z^* = 0$  attractor.

If  $p^0 > 0$  the two terms in Eq. (4.13) sum up into the following expression ( $p^0 > 0, q_1 < 0, q_2 < 0, q_3 > 0$ ):

$$Z^* = \frac{i}{\sqrt{2}} (p^0 q_1 q_2 q_3)^{1/4} \neq 0. \quad (4.19)$$

Thus, the BH charge configuration  $p^0 > 0, q_1, q_2 < 0$ , and  $q_3 > 0$  supports a  $d = 4$  non-BPS  $Z^* \neq 0$  attractor, whose entropy is given by Eq. (4.17). Thus,

concerning the  $d = 4$  supersymmetry-preserving features, the two BH charge configurations  $(p^0 > 0, q_1 > 0, q_2 > 0, q_3 > 0)$  (upliftable to  $d = 5$  BPS) and  $(p^0 > 0, q_1 < 0, q_2 < 0, q_3 > 0)$  (upliftable to  $d = 5$  non-BPS) are *equivalent*.

Summarizing, we have found that the BH charge configuration specified by the charge vector  $Q_e = (p^0, 0, 0, q_i)$  splits into three *inequivalent* configurations, depending on the signs of the charges. The critical value of the  $d = 4$  effective BH potential has the general form

$$V_{(p^0, q_i), crit.}^* = 2 |p^0 q_1 q_2 q_3|^{1/2}, \quad (4.20)$$

but the  $\mathcal{N} = 2$ ,  $d = 4$  central charge correspondingly takes three different values:

$$\begin{aligned} p^0 < 0, q_1 > 0, q_2 > 0, q_3 > 0 &\Leftrightarrow \frac{1}{2} - BPS : |Z|_{\frac{1}{2}-BPS} = \sqrt{2} (-p^0 q_1 q_2 q_3)^{1/4}; \\ p^0 < 0, q_1 < 0, q_2 < 0, q_3 > 0 &\Leftrightarrow non - BPS, Z = 0 : Z_{non-BPS, Z=0} = 0; \\ \left. \begin{aligned} p^0 > 0, q_1 > 0, q_2 > 0, q_3 > 0 \\ p^0 > 0, q_1 < 0, q_2 < 0, q_3 > 0 \end{aligned} \right\} &\Leftrightarrow non - BPS, Z \neq 0 : \begin{cases} |Z|_{non-BPS, Z \neq 0} = \\ = \frac{1}{\sqrt{2}} (p^0 q_1 q_2 q_3)^{1/4}. \end{cases} \end{aligned} \quad (4.21)$$

It is useful to point out that in [48] a different normalization for the  $d = 5$  (*electric*) effective BH potential was used:

$$V_5^{BH} = 3V_5^q, \quad (4.22)$$

due to a different normalization of the  $d = 5$  vector kinetic matrix:  $\overset{\circ}{a}_{ij}^V = \frac{1}{3}a_{ij}$ . As a consequence

$$V_{5, crit.}^{BH} = 9 |q_1 q_2 q_3|^{2/3}, \quad (4.23)$$

and the corresponding  $d = 5$  entropy is given by Eq. (3.15). Since  $Z_{5, BPS}^q = 3 |q_1 q_2 q_3|^{1/3}$  and  $Z_{5, non-BPS}^q = |q_1 q_2 q_3|^{1/3}$ , one then retrieves the  $d = 5$  *sum rules* derived in [48], *i.e.*

$$V_{5, BPS}^{BH} = Z_{5, BPS}^2, \quad V_{5, non-BPS}^{BH} = 9 Z_{5, non-BPS}^2, \quad (4.24)$$

which also shows that

$$\frac{3}{2} [g^{xy} (\partial_x Z_5) \partial_y Z_5]_{non-BPS} = 8 Z_{5, non-BPS}^2, \quad (4.25)$$



due to the identity [48]

$$V_5^{BH} = Z_5^2 + \frac{3}{2} g^{xy} (\partial_x Z_5) \partial_y Z_5, \quad (4.26)$$

where the  $d = 5$  scalar metric in our convention reads

$$g_{xy} = \frac{1}{2} \left( \partial_x \hat{\lambda}_i \right) \left( \partial_y \hat{\lambda}_j \right) a^{ij}, \quad (4.27)$$

with  $\hat{\lambda}_i \equiv a_{ij} \hat{\lambda}^j$ .

## 5 Conclusion

In this study we have related the  $d = 4$  and  $d = 5$  entropy formulæ for special geometry described by a cubic holomorphic prepotential function ( $d$ -geometry), based solely on properties of the general black hole effective potential (2.13), rather than considering solutions for the scalar fields. These  $d$ -geometries are particularly relevant, as they describe the large volume limit of the  $CY_3$ -compactifications of type IIA superstrings or, in the  $d = 5$  uplift, the  $CY_3$ -compactifications of  $M$ -theory [70].  $d$ -geometries also include homogeneous and symmetric special geometries, where we have shown that they have even more interesting properties.

It is now worth commenting about the charge configurations used in our treatment. For symmetric  $d$ -geometries the configurations  $(p^0, 0, 0, q_i)$  (*electric*, upliftable to  $d = 5$  extremal Tangherlini BH with horizon geometry  $AdS_2 \times S^3$ ) and  $(0, p^i, q_0, 0)$  (*magnetic*, upliftable to  $d = 5$  black string with horizon geometry  $AdS_3 \times S^2$ ) are reciprocally dual. For the electric configuration  $(p^0, 0, 0, q_i)$  (and for vanishing axions) the  $d = 5$  and  $d = 4$  effective BH potentials are respectively denoted by  $V_5^q$  and  $V^*$ , and respectively given by Eqs. (3.6) and (3.5). For the magnetic configuration  $(0, p^i, q_0, 0)$  (and for vanishing axions) the  $d = 5$  and  $d = 4$  effective BH potentials are respectively denoted by  $V_5^p$  and  $V^*$ , and respectively given by

$$V_5^p = a_{ij} p^i p^j = V_5^p \left( \hat{\lambda}^i, p^j \right); \quad (5.1)$$

$$V^* \left( \hat{\lambda}^i, \mathcal{V}, p^j, q_0 \right) = \frac{1}{2} \left[ \mathcal{V}^{-1} (q_0)^2 + \mathcal{V}^{1/3} V_5^p \left( \hat{\lambda}^i, p^j \right) \right]. \quad (5.2)$$

By comparing Eqs. (5.1) and (5.2) with Eqs. (3.6) and (3.5), it is easy to realize that such two pairs of Eqs. are related (in the  $stu$  model, thus

$i = 1, 2, 3$ ) by the transformations

$$\begin{aligned}\mathcal{V} &\longleftrightarrow \mathcal{V}^{-1}; \\ q_i &\longleftrightarrow p^i; \\ |q_0| &\longleftrightarrow |p^0|; \\ \widehat{\lambda}^i &\longleftrightarrow \left(\widehat{\lambda}^i\right)^{-1}.\end{aligned}\tag{5.3}$$

The critical values of the potentials given by Eqs. (5.1) and (5.2) respectively are

$$\frac{V_{5,crit.}^p}{3} = \left| \frac{1}{3!} d_{ijk} p^i p^j p^k \right|^{2/3} = |p^1 p^2 p^3|^{2/3};\tag{5.4}$$

$$V_{crit.}^* (p^j, q_0) = 2 |q_0 p^1 p^2 p^3|^{1/2}.\tag{5.5}$$

Moreover, the  $d = 5$  volume is related to  $V_5^p$  as follows:

$$\mathcal{V}^{-4/3} = \frac{1}{3(q_0)^2} V_5^p,\tag{5.6}$$

yielding the critical value

$$\mathcal{V}_{cr.}^{-4/3} = \frac{1}{(q_0)^2} \left( \frac{1}{3!} |d_{ijk} p^i p^j p^k| \right)^{2/3}.\tag{5.7}$$

Notice that Eqs. (5.6) and (5.7) can respectively be obtained from Eqs. (3.8) and (3.12) by applying the mapping (5.3). The  $d = 5$  and  $d = 4$  supersymmetry-preserving features for the magnetic configuration goes the same way as for the electric configuration, with the interchanges  $p^0 \longrightarrow -q_0$  and  $q_i \longrightarrow p^i$  ( $i = 1, 2, 3$ ).

We remark that Eqs. (5.4) and (5.7) (as well as Eqs. (5.1) and (5.2)) are valid also for non-symmetric  $d$ -geometries, because they make use of the completely covariant tensor  $d_{ijk}$  (the same appearing in the holomorphic prepotential). However, let us stress that for non-symmetric  $d$ -geometries the (relevant expressions for the) electric configuration is much more complicated, due to the lack of a globally constant tensor  $d^{ijk}$  (which in the non-symmetric case does not satisfy the relation (2.15)).

It would be intriguing to study the “flat” directions of the effective BH potentials for (symmetric and non-symmetric)  $d = 5$  (special real) and  $d = 4$  (special Kähler)  $d$ -geometries, and relate the corresponding moduli spaces [43] through the results obtained in the present work.

Here we limit ourselves to observe that the moduli space of  $d = 5$  non-BPS attractors is both a submanifold of the moduli space of the  $d = 4$  non-BPS  $Z \neq 0$  attractors (coinciding with the corresponding  $d = 5$  special real

manifold) and of the moduli space of the  $d = 4$  non-BPS  $Z = 0$  attractors. This non-trivial result, obtained in [43], can also be understood as an outcome of our analysis of this paper where, for the so-called electric - and magnetic - charge configurations, we have shown that the  $d = 5$  non-BPS attractors can give rise to both classes ( $Z \neq 0$  and  $Z = 0$ ) of  $d = 4$  non-BPS attractors. Consequently, the  $d = 5$  non-BPS “flat” directions must be common to both types of  $d = 4$  non-BPS “flat” directions.

It would be interesting to analyze these issues in more detail, and understand better the “flat” directions (and their fate once the quantum corrections are switched on), as they depend on the background charge configuration vector  $Q$ .

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